LAMINAR FLOW AND PRESSURE DROP IN INTERNALLY FINNED ANNULAR DUCTS

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INTERNALLY FINNED annular flow passages have seen service in conventional heat-exchanger applications for some time [1] and have recently been considered in the design of fin-tube radiators for space vehicles [2]. Such flow passages are pictured schematically in the inset of Fig. 1, which shows an annular space subdivided by radial fins. The fins extend without interruption along the axial length of the duct.

Pressure-drop information for internally finned annular ducts appears to be rather limited in extent. Indeed, the present authors are aware of only a single prior investiga- of the problem, that one being confined to measurements in a duct with radius ratio $r_1/r_2 = 0{\cdot}65$ and opening angles θ_0 in the range 8° to 13° [1]. Friction-factor results, presumably corresponding to fully developed flow conditions, are reported for both the laminar regime and the low Reynolds-number portion of the turbulent regime. The laminar data appear to correlate satisfactorily with the circular-tube friction factor when the equivalent diameter is used as the characteristic dimension in the Reynolds number. The latter finding is somewhat surprising inasmuch as the use of the equivalent diameter is generally insufficient for correlating laminar-flow friction factors for non-circular and circular ducts.

The aim of the present investigation is to obtain a complete analytical solution for the fully developed laminar flow and pressure drop in internally finned annular flow passages. The starting point of the analysis is the momentum equation for fully developed laminar flow

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{1}{\mu}\frac{\mathrm{d}p}{\mathrm{d}z} = \text{constant} \qquad (1)$$

in which u is the axial velocity, z the corresponding axial coordinate, and p the static pressure. The radial coordinate r and the angular coordinate θ are pictured in Fig. 1. For purposes of analysis, consideration may be confined to a single flow passage bounded between the circular arcs $r = r_1$ and $r = r_2$ and the rays $\theta = 0$ and $\theta = \theta_0$. The velocity must be zero on these boundaries in accordance with the no-slip condition.

The velocity solution is facilitated by introducing a difference velocity u^* defined by

$$u^* = u + (r^2/4\mu)\,(-\mathrm{d}p/\mathrm{d}z) \qquad (2)$$

It is easily verified that u^* must obey Laplace's equation. In addition, u^* takes on the values $(r^2/4\mu)\,(-\mathrm{d}p/\mathrm{d}z)$ on the boundaries. Then, by introducing the transformation of coordinates $X = \ln r/r_1$, $\theta = \theta$ the annular sector is transformed into a rectangular region $0 \leqslant X \leqslant \ln r_2/r_1$, $0 \leqslant \theta \leqslant \theta_0$, and the governing equation for u^* becomes

$$\partial^2 u^*/\partial X^2 + \partial^2 u^*/\partial \theta^2 = 0 \qquad (3)$$

On the edges of the rectangle, the boundary values of $u^*/[(r_1^2/4\mu)\,(-\mathrm{d}p/\mathrm{d}z)]$ are: 1 at $X = 0$, $(r_2/r_1)^2$ at $X = \ln r_2/r_1$, e^{2X} at $\theta = 0$ and $\theta = \theta_0$.

The solution of equation (3) subject to the above-mentioned boundary conditions is conveniently carried out by the method of separation of variables and the use of linear superposition. Once the u^* has been solved for, it may be combined with equation (2); from this, there follows the complete solution for the velocity distribution in the duct.

$$\frac{u}{\dfrac{r_1^2}{4\mu}\left(-\dfrac{\mathrm{d}p}{\mathrm{d}z}\right)} = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(1/n)}{1+\left(\dfrac{2\alpha}{n\pi}\right)^2}\left[1-(-1)^n\left(\frac{r_2}{r_1}\right)^2\right]$$
$$\cdot\frac{\cosh n\pi/\alpha\,(\theta_0/2-\theta)}{\cosh n\pi/\alpha\ \theta_0/2}\sin\frac{n\pi X}{\alpha} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}[1-(-1)^n]$$
$$\cdot\left[\frac{\sinh n\pi/\theta_0\,(\alpha - X) + (r_2/r_1)^2\sinh n\pi X/\theta_0}{\sinh n\pi\alpha/\theta_0}\right]\sin\frac{n\pi\theta}{\theta_0} - (r/r_1)^2 \qquad (4)$$

in which

$$X = \ln r/r_1,\quad \alpha = \ln r_2/r_1 \qquad (5)$$

The velocity at any position r, θ may be found by numerical evaluation of equation (4).

The volume flow-rate Q passing through any cross section may be determined by integrating the velocity distribution

$$Q = \int_0^{\theta_0}\int_{r_1}^{r_2} ur\,\mathrm{d}r\,\mathrm{d}\theta. \qquad (6)$$

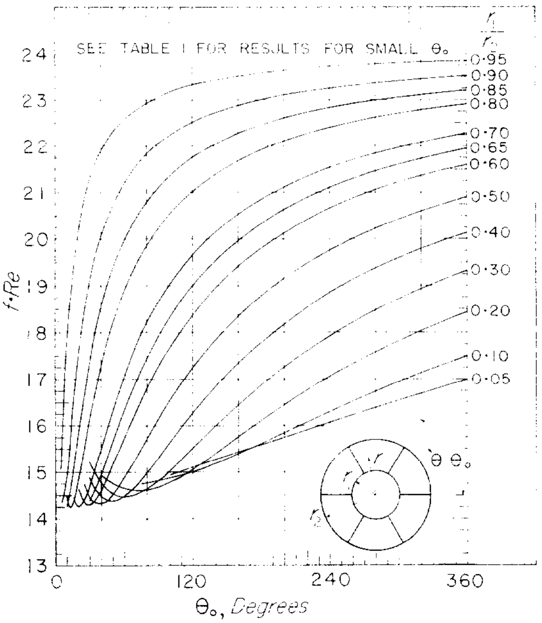

FIG. 1. Friction factor results for internally-finned annular ducts.

Upon substituting and carrying out the indicated operations, one finds

$$\frac{Q}{\frac{r_1^4}{4\mu}\left(-\frac{dp}{dz}\right)} = \frac{\theta_0}{4}\left[1-\left(\frac{r_2}{r_1}\right)^4\right] + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(n\pi/\alpha)^2\,[1-(-1)^n\,(r_2/r_1)^2]^2}{n\,[4+(n\pi/\alpha)^2]^2}\tanh\frac{n\pi}{\alpha}\frac{\theta_0}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(1/n)[1-(-1)^n]^2}{[4-(n\pi/\theta_0)^2]\sinh n\pi\alpha/\theta_0}\left\{\frac{2\theta_0}{n\pi}\left[\left(\frac{r_2}{r_1}\right)^4-1\right]\sinh\frac{n\pi\alpha}{\theta_0} - \left[\left(\frac{r_2}{r_1}\right)^4+1\right]\cosh\frac{n\pi\alpha}{\theta_0} + 2\left(\frac{r_2}{r_1}\right)^2\right\} \quad (7)$$

The foregoing provides the relationship between the volume flow and the pressure drop. Alternate forms of equation (7) may also be obtained by introducing the mass flow rate $\dot{m} = \rho Q$ or the mean velocity $\overline{U} = Q/A$, wherein ρ and A are respectively the density and cross-sectional area.

It is customary to present the pressure drop results in terms of a friction factor f defined as

$$f = \frac{(-\,dp/dz)}{0{\cdot}5\rho\overline{U}^2}\frac{D_e}{4} \quad (8)$$

in which D_e is the equivalent diameter. Upon rearranging the definition of f, there is obtained

$$f\cdot Re = \left[\frac{(r_1^4/4\mu)(-\,dp/dz)}{Q}\right]\frac{2AD_e^2}{r_1^4}. \quad (9)$$

The quantity in brackets is the reciprocal of the flow-pressure drop relationship given by equation (7), while

$$Re = \frac{\rho\overline{U}D_e}{\mu},\quad \frac{A}{r_1^2} = \frac{\theta_0}{2}\left[\left(\frac{r_2}{r_1}\right)^2 - 1\right],$$
$$\frac{D_e}{r_1} = \frac{2\theta_0\,[(r_2/r_1)^2 - 1]}{\theta_0\,(r_2/r_1+1) + 2\,(r_2/r_1-1)}. \quad (10)$$

By inspection of the foregoing, it is seen that $f\cdot Re$ depends upon two geometrical parameters, r_2/r_1 and θ_0.

The $f\cdot Re$ product has been numerically evaluated for a wide range of values of the governing parameters. These results have in part been plotted in Fig. 1 ahd are also listed in Table 1 for the smaller values of the opening angle θ_0. From this presentation, it is seen that $f\cdot Re$ ranges from about 14 to 24. On the other hand,

Table 1. $f\cdot Re$ results of range of smaller θ_0

	$\theta_0 \rightarrow$	5°	10°	15°	20°	30°	40°	50°	60°
$r_1/r_2 =$	0·05	13·42	13·55	13·68	13·81	14·03	14·22	14·38	14·52
	0·1	14·47	14·49	14·51	14·54	14·59	14·62	14·66	14·69
	0·2	16·34	16·04	15·79	15·57	15·21	14·94	14·75	14·65
	0·3	17·80	17·07	16·47	15·98	15·24	14·78	14·54	14·47
	0·4	18·75	17·51	16·56	15·84	14·90	14·48	14·38	14·49
	0·5	19·14	17·37	16·13	15·30	14·47	14·33	14·52	14·88
	0·6	18·94	16·68	15·35	14·64	14·29	14·60	15·14	15·72
	0·65	18·60	16·16	14·91	14·38	14·40	14·97	15·65	16·32
	0·7	18·08	15·56	14·52	14·26	14·71	15·52	16·33	17·04
	0·8	16·42	14·42	14·33	14·84	16·15	17·28	18·16	18·86
	0·85	15·32	14·26	14·91	15·82	17·39	18·52	19·34	19·95
	0·9	14·35	14·98	16·36	17·51	19·07	20·05	20·71	21·18
	0·95	15·06	17·62	19·17	20·14	21·26	21·88	22·28	22·56

$f \cdot Re = 16$ for the circular tube. It follows, therefore, that the tube results are not applicable to internally finned annular ducts of arbitrary geometrical configuration. However, it may be noted that for conditions corresponding to the experiments of reference [1], the present analysis predicts $f \cdot Re \sim 15{\cdot}5$–$17{\cdot}5$. It is thus seen that there are special situations in which the results for the finned annulus are close to those of the circular tube. As chance would have it, just such a case was studied in reference [1].

It is interesting to note that the $f \cdot Re$ results are not monotonic functions of the governing parameters, especially at smaller opening angles θ_0. The explanation for this lies in the fact that $f \cdot Re$ not only includes flow-related quantities, but also includes geometrical quantities. As the parameters are varied, some of these quantities may increase and others may decrease, thereby providing the possibility of maxima and minima.

REFERENCES

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